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Dynamics of elastic bodies prestressed by internal slipping cables

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Abstract

This paper examines a system consisting of a deformable body with a slipping cable in its interior. The cable may be employed both as an actuator and as a sensor for the body, thanks to the particular coupling arising between local cable strain and global body deformation provided by the cable slip. The system is analyzed by interpreting the coupling as a constraint with global nature exerted by the deformable body on the cable deformation. The descriptors of the reduced kinematics of the system are established and the formulation is developed according to the exact deformation theory. Successively, the problem is linearized in proximity of a known solution and the particular case of homogeneous, massless cable is presented. A simple but meaningful application is also reported. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

One of the possible techniques for obtaining a control on the strain or stress state of a deformable elastic solid consists of introducing at its interior a cable anchored at the ends but free to slip along a path which remains linked to the body. The possibility of controlling the cable traction force, e.g. by means of thermoelastic or piezoelectric effects or through a direct control on deformation by extracting a portion of the cable from one anchorage, permits the production of prefixed stress and strain fields, whose characteristics are ruled by the geometry of the path followed by the cable. On the other hand, whenever it is possible to measure the cable traction, the coupled system can be employed for obtaining information on the body deformation. The coupled system can therefore be employed both as an

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Nomenclature

a	cable stetch
$\mathscr{B}$	body
$\mathbf{b}$	mass force in the body
$\mathscr{C}$	cable
$\mathbf{E}$	tangent to the cable $\mathscr{C}$ in the reference configuration
e	deformation of $\mathbf{E}$
$\bar{\mathbf{e}}$	tangent to the deformed cable
$\mathbf{f}$	surface traction on the body $\mathscr{B}$
$\mathbf{G}$	tangent to the curve $\mathscr{H}$ in the reference configuration
$\mathbf{g}_{\mathrm{p}}$	deformation of $\mathbf{G}$
$\bar{\mathbf{g}}_p$	tangent to the deformed curve
$\mathscr{H}$	curve traced on the body
$\mathbf{H}$	curve $\mathscr{H}$ in the reference configuration
$\mathbf{h}_{\mathrm{p}}$	deformed curve $\mathscr{H}$
L	length of the cable $\mathscr{C}$ and curve $\mathscr{H}$ in the reference configuration
l_{p}	length of the deformed curve $\mathscr{H}$
$\mathbf{P}$	position of the body points in the reference configuration
$\mathbf{p}$	deformation of the body $\mathscr{B}$
$\mathbf{R}$	position of the cable points in the reference configuration
$\mathbf{r}$	deformation of the cable $\mathscr{C}$
$\mathbf{S}$	first Piola–Kirchhoff stress tensor
$\bar{\mathbf{S}}$	second Piola–Kirchhoff stress tensor
$\mathbf{T}$	Cauchy stress tensor
$\mathbf{t}$	traction force on the cable $\mathscr{C}$
$\mathbf{u}$	displacement field of the body $\mathscr{B}$
w	strain energy density of the body $\mathscr{B}$
X_i	material co-ordinate of the body points
α	descriptor of the constrained cable deformation
β	mass force in the cable
η	curvilinear abscissa of the curve $\mathscr{H}$
ρ	material cable points
ξ	descriptor of the constrained cable displacements
ω	strain energy density of the body $\mathscr{C}$

actuator and a sensor, and the applications are of practical interest in structural mechanics, robotics, biomechanics and measurement techniques.

The cable slipping leads to a particular coupling relating the local cable strain to the global body deformation. This forms the most characteristic aspect of the system's behaviour and permits us to provide or measure the effects on the whole body by means of control or measurement of the cable state at one point only.

The existing literature on this topic mainly regards simple and very specific problems of structural engineering (e.g. Naaman and Alkhairi, 1991; Alkhairi and Naaman, 1993) Formulations in a more general context are presented in Dall'Asta (1995) and Dall'Asta and Leoni (1997). These latter studies are dedicated to the static case and consider a homogeneous cable free from mass forces. In this case, cable strain is uniform and its kinematics can be entirely deduced from the sole three-dimensional body deformation.

This paper intends to analyze a more general situation, where the cable is no longer homogeneous and may undergo mass forces related to acceleration or external fields. Here, the cable is not homogeneous and its kinematics are no longer deducible from the body deformation, even if it is still subjected to a constraint with a global nature, due to the fact that the cable path is linked to the body. This leads to richer kinematics and makes it necessary to define new entities for its description, so that the treatment becomes substantially different and more complex with respect to that previously analyzed.

In particular, the system is analyzed by assuming a three-dimensional continuum model for the body while the cable is schematized by means of a uni-dimensional model. The coupling between the components is translated into analytical form by a condition of global constraint (see Antman and Marlow, 1991) which reduces the set of possible deformations of the cable on the basis of the body deformation and the initial geometry of the path. An attempt is made to furnish a convenient representation of the constraint deformation by introducing a unique independent scalar valued function having the rule of kinematic descriptor of the cable and furnishing its deformation. The system description continues by stating the balance conditions on the basis of the D'Alembert principle. This permits establishing the dynamical entities which are the dual of the descriptors of the cable deformation. A local interpretation of the results evidences some characteristic aspects of the reactive forces arising at the interface between cable and body. Finally, the linearized theory obtained from the complete theory by assuming that the motion develops in a sufficiently small neighbourhood of the reference configuration, is described. Beyond the practical interest, the linear theory simplifies some aspects of the coupling between slipping cable and body and can be useful for the comprehension of the system behaviour.

Successively, the case of an homogeneous massless cable is examined. In this case, the set of the constrained deformation is reduced further and, even if it may be deduced from the previous as a particular case, it is convenient to develop a slightly different and autonomous formulation.

Finally, a simple applicative example concerning the effects of an internal stretched cable on the eigenfrequencies of a plate is described.

2. Kinematics

The analyzed system consists of a solid deformable body and a cable; their coupling is obtained by linking the cable ends to two points of the body and by constraining the cable to follow a path provided by a curved tunnel in the interior of the body (Fig. 1); the cable can slip along its path. In this paper, a description of the system is presented by modeling the cable as a uni-dimensional manifold embedded into the remaining part of the system modeled as a three-dimensional manifold.

The particles of the body $\mathscr{B}$, consisting of the deformable solid and its interior tunnel, are identified by means of the three material co-ordinates (X_i; $i = 1,2,3$) with respect to the orthonormal basis $\{\mathbf{A}_i\}$ which localize their positions $\mathbf{P}(X_i) = X_i\mathbf{A}_i$ in the reference configuration, letting Ω be the domain $\mathbf{P}(\mathscr{B})$ occupied by the body.

Furthermore, let $\mathscr{H} = \{X_i = H_i(\eta);\ \eta \in I_\eta = [0,s]\}$ be a subset of $\mathscr{B}$ which describes the regular curve $\mathbf{H}(\eta) = H_i(\eta)\mathbf{A}_i$ in the reference configuration. This curve models the axis of the tunnel in which the cable is disposed and along which it can slip. In reality, both these entities, the cable and its path, are three-dimensional but they have one dimension sufficiently greater than the others to justify the uni-dimensional idealization. The tangent vector $\mathbf{H},_\eta$ is defined everywhere, its modulus $|\mathbf{H},_\eta|$ is always positive and the unit tangent vector is denoted by $\mathbf{G}(\eta) = \mathbf{H},_\eta/|\mathbf{H},_\eta|$ (commas denote derivatives with respect to spatial variables). The indexes O and S are used for labeling the values assumed by a generic quantity at the two material points (X_{Oi}), (X_{Si}), located at the ends of the curve. In the reference configuration, these points occupy the positions $\mathbf{P}_O = \mathbf{H}(0) = X_{Oi}\mathbf{A}_i$, $\mathbf{P}_S = \mathbf{H}(s) = X_{Si}\mathbf{A}_i$.

The cable $\mathscr{C}$ is a uni-dimensional manifold whose material points are identified by means of the material co-ordinate $\rho \in I_\rho = [0,s]$ and the cable describes the curve $\mathbf{R}(\rho) = R_i(\rho)\mathbf{A}_i$ in the reference configuration. In this configuration, the curves $\mathbf{R}$ and $\mathbf{H}$ coincide in the sense that the cable particle labeled by $\zeta \in [0,s]$ occupies the same position of the body material point $(H_i(\zeta))$ lying on the curve and labeled by the same curvilinear abscissa, i.e.

$$\mathbf{R}(\zeta) = \mathbf{H}(\zeta). \ \zeta \in [0,s] \tag{1}$$

Consequently, the cable ends occupy the same positions of the body particles (X_{Oi}) and (X_{Si}). It may be useful to remark that the choice of assuming ρ and η to define the same position in the reference configuration derives from the observation that the most convenient way to express a parametrized space curve is usually unique; it should however not be forgotten that $\rho \in I_\rho$ and $\eta \in I_\eta$ are two substantially different quantities because the former denotes a material particle of the cable, while the latter is a parameter denoting a material particle of the body, through the functions $H_i(\eta)$. From this, it follows that the previous correspondence will be lost in configurations which are different from that assumed as reference. The function $\mathbf{E}(\rho)$ denotes the unit tangent vector $\mathbf{R}_{,\rho}/|\mathbf{R}_{,\rho}|$, coinciding with $\mathbf{G}$ for $\eta = \rho$, and the length of the cable from its former anchorage point $\mathbf{P}_O$ to the generic material particle ρ is described by means of the function

$$\Lambda(\rho) = \int_0^\rho |\mathbf{R}_{,\zeta}| \mathrm{d}\zeta. \tag{2}$$

The motion of the system components is described by the two functions $\mathbf{p}(X_k;t) = p_i(X_k;t)\mathbf{A}_i$ and $\mathbf{r}(\rho;t) = r_i(\rho;t)\mathbf{A}_i$ which furnish the positions of the body $\mathscr{B}$ and the cable $\mathscr{C}$ at a generic instant t by starting from the reference configuration. Hereinafter, it is assumed that the field $\mathbf{p}$ is compatible with the body constraints.

The deformation description in the neighborhood of a body material point is furnished by the quantity $\nabla\mathbf{p}(X_k;t) = p_{i,j}(X_k;t)\mathbf{A}_i \otimes \mathbf{A}_j$, velocity and acceleration are denoted by $\dot{\mathbf{p}}(X_k;t) = \dot{p}_i(X_k;t)\mathbf{A}_i$ and $\ddot{\mathbf{p}}(X_k;t) = \ddot{p}_i(X_k;t)\mathbf{A}_i$ (dots denote partial derivatives with respect to time). It is here assumed that $\mathbf{p}$ is orientation-preserving and locally invertible, at least almost everywhere, on Ω, i.e. $\det(\nabla\mathbf{p}) > 0$.

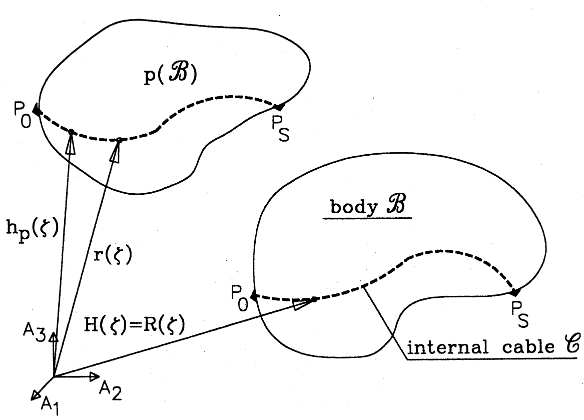

Fig. 1. Cable–body system.

Furthermore, the body motion is sufficiently regular in the neighbourhood of $\mathscr{H}$ to permit defining its trace $\mathbf{p}(\mathscr{H};t)$ on the curve $\mathscr{H}$ such that $|\nabla\mathbf{p}\mathbf{H},_\eta| > 0$. The complex question of global invertibility is beyond the scope of this paper.

The motion $\mathbf{p}(\mathscr{H};t)$ of the curve $\mathscr{H}$ rigidly linked to the body can be deduced through the functions $\mathbf{H}_i$ and the notation $\mathbf{h}_p(\eta)=\mathbf{p}(H_i(\eta);t)$ will be used for denoting the vector function describing this deformed curve in correspondence with the body deformation $\mathbf{p}$ present at the instant t. The derivatives with respect to the parameter η can also be related to the body deformation by means of the relations

$$\mathbf{h}_{p,\eta}(\eta) = \mathbf{g}_p(\eta) = \nabla\mathbf{p}(H_i(\eta);t)\mathbf{H},_\eta(\eta) \tag{3}$$

and

$$\mathbf{h}_{p,\eta\eta}(\eta) = \nabla\nabla\mathbf{p}(H_i(\eta);t)(\mathbf{H},_\eta(\eta)\otimes\mathbf{H},_\eta(\eta)) + \nabla\mathbf{p}(H_i(\eta);t)\mathbf{H},_{\eta\eta}(\eta). \tag{4}$$

Its modulus $|\mathbf{h}_{p,\eta}|$ is assumed to be positive and the unit tangent vector $\bar{\mathbf{g}}_p(\eta) = \mathbf{h}_{p,\eta}/|\mathbf{h}_{p,\eta}|$ is defined. The second derivative, $\mathbf{h}_{p,\eta\eta}$ has a component in the direction of the curve normal, proportional to the actual curvature of the path. The velocity and acceleration of the body material points lying on the curve will be denoted by

$$\dot{\mathbf{h}}_p(\eta) = \dot{\mathbf{p}}(H_i(\eta);t) \tag{5}$$

and

$$\ddot{\mathbf{h}}_p(\eta) = \ddot{\mathbf{p}}(H_i(\eta);t), \tag{6}$$

while the time derivative of $\mathbf{H},_\eta$ will be denoted by

$$\dot{\mathbf{h}}_{p,\eta}(\eta) = \nabla\dot{\mathbf{p}}(H_i(\eta);t)\mathbf{H},_\eta(\eta). \tag{7}$$

It is useful for the following description of the cable constraint to evaluate the length σ of the curve from its origin at $\mathbf{P}_O$ to the generic point $(H_i(\eta))$. This length is furnished by

$$\sigma = \gamma_p(\eta) = \int_0^\eta |\mathbf{h}_{p,\zeta}|\mathrm{d}\zeta, \tag{8}$$

where γ_p: $I_\eta \rightarrow [0,l_p]$ is an invertible function related to the actual deformation and l_p denotes the total length of the curve in the actual configuration. From this, it can also be deduced that its inverse, γ_p^{-1}:$[0,l_p]\rightarrow I_\eta$, also exists and $\gamma_p^{-1}(\sigma)$ provides the curvilinear co-ordinate η of the point which is located at the end of a curve tract with assigned length σ. At each instant t, a different function γ_p is defined as a consequence of the body motion $\mathbf{p}$.

A description of the local deformation of the cable can be furnished by the vector valued function $\mathbf{r},_\rho(\rho;t)$ or, equivalently, by means of $\mathbf{e}(\rho;t)=\mathbf{r},_\rho/|\mathbf{R},_\rho|$ which identifies the vector provided by the transformation of the tangent vector $\mathbf{E}(\rho)$. The local elongation is measured by means of the scalar quantity $a(\rho;t)=|\mathbf{r},_\rho|/|\mathbf{R},_\rho|$, so that, by introducing the unit tangent vector in the deformed configuration, $\bar{\mathbf{e}}(\rho;t) = \mathbf{r},_\rho/|\mathbf{r},_\rho|$, the local deformation can also be written in the form $\mathbf{r},_\rho := a|\mathbf{R},_\rho|\bar{\mathbf{e}}$. The same parametrization with respect to the length, previously used for the curve $\mathscr{H}$, can now be introduced at the instant t in the form

$$\sigma = \lambda_r(\rho) = \int_0^\rho a|\mathbf{R},_\zeta|\mathrm{d}\zeta, \tag{9}$$

where λ_r: $I_\rho \longrightarrow [0,l_r]$ is an invertible function related to the cable deformation $\mathbf{r}$ active at the instant t and l_r is the actual total length of the cable.

It is evident that not all the deformations $\mathbf{r}(\rho;t)$ are suitable for describing the cable motion since it must remain within the curve $\mathbf{h}_p(\eta)$, even if its material points are free to slip along that curve, consistently with the assumption of local invertibility. This situation introduces a global constraint on $\mathbf{r}$ (Antman and Marlow, 1991) which requires that the co-domains $\mathbf{p}(\mathscr{H};t)$ and $\mathbf{r}(\mathscr{C};t)$ coincide and cannot be reduced to a local constraint prescribing simple restrictions on the local deformation. These latter assertions are intuitive statements that must be translated into a more precise condition on $\mathbf{r}$ providing the manifold of admissible deformations. The constraint condition for $\mathbf{r}$ can be obtained by taking advantage of the two parametrizations with respect to the curve length and requiring that each cable material point ρ tracing an arc with length $\sigma = \lambda_r(\rho)$, occupies the position of that body point lying on the curve $\mathscr{H}$ and labeled by $\eta = \gamma_p^{-1}(\sigma)$, which is related to an arc on $\mathscr{H}$ with the same length, i.e.

$$\mathbf{r}(\rho;t) - \left(\mathbf{h}_p \bigcirc \gamma_p^{-1} \bigcirc \lambda_r\right)(\rho) = 0 \;\; \rho \in I_\rho. \tag{10}$$

This also implies that $l_r = l_p$ and furnishes a prescription which varies in time as a consequence of the body motion $\mathbf{p}$. The previous condition on deformations can be replaced by an alternative condition on the local deformation measures by means of Eq. (9) and by observing that $\gamma_p^{-1},_\lambda = \gamma_{p},_\eta$. It relates $\mathbf{r},_\rho$ to the strain measure a and the curve unit tangent vector $\bar{\mathbf{g}}_p(\eta) = \mathbf{g}_p/|\mathbf{g}_p|$ by means of the following relations

$$\mathbf{r},_p(\rho;t) - a(\rho;t)|\mathbf{R},_\rho|\left(\bar{\mathbf{g}}_p \bigcirc \gamma_p^{-1} \bigcirc \lambda_r\right)(\rho) = 0 \tag{11}$$

and

$$\mathbf{r}(0;t) - \mathbf{h}_p(0) = 0. \tag{12}$$

This differential form shows that the local strain measure a of the cable is not related to the deformation of its neighborhood but depends on the global deformation of the body along the path via γ_p^{-1} and l_r and cannot be expressed by an algebraic law but requires a functional dependence also involving the deformation $\mathbf{p}$ and the initial geometry described by $\mathbf{H}$.

Both the previous expressions given to the constraint permit formulating the problem by means of Lagrange multipliers, even though it is more convenient to obtain a representation of the admissible cable deformation in order to reduce the dimension of the problem. Eq. (10) may be useful for this scope, simply by observing that $\mathbf{h}_p$ and γ_p^{-1} derive from the body deformation while each invertible function $\lambda(.;t)$: $I_\rho \longrightarrow [0,l_p]$ may replace the particular function λ_r to furnish a deformation $\mathbf{r}$ which is admissible at the instant t. The set of invertible functions from I_ρ to $[0,l_p]$ provides all the functions $\mathbf{r}$ satisfying the constraint and can be adopted to represent the constrained kinematics of the cable. This is not however profitable because deriving the function γ_p^{-1} from the curve deformation $\mathbf{h}_p$ is often complex, while it is more convenient to describe the cable deformation $\mathbf{r}$ at the instant t by means of the function $\alpha(.,t) = \gamma_p^{-1} \bigcirc \lambda$: $I_\rho \rightarrow I_\eta$ which is equally a generic invertible function ($\alpha,_\rho > 0$), thanks to the property of γ_p^{-1}, and permits us to represent admissible cable deformations in the form

$$\mathbf{r}(\rho;t) = (\mathbf{h}_p \bigcirc \alpha)(\rho;t) = \mathbf{h}_p(\alpha)(\rho;t) = \mathbf{p}(H_i(\alpha(\rho;t));t) \tag{13}$$

without having to evaluate the inverse of γ_p. The relation $\eta = \alpha(\rho;t)$ establishes a relation between the points of $\mathscr{C}$ and $\mathscr{H}$ which occupies the same spatial position at the instant considered and describes the slipping $\alpha(\rho;t) - \rho$ of the cable points along the path.

In conclusion, the system's motion is completely defined by the two functions α and $\mathbf{p}$ so that its kinematics, and in particular the kinematics of the constrained cable, can be described with reference to these two descriptors only. The nonlinear dependence Eq. (13) of the cable deformation on both the kinematical descriptors is somewhat involved and provides a strict coupling between body deformation, cable slipping and path geometry. Hereinafter, we will attempt to make the dependence of generic quantities on these kinematic descriptors $\mathbf{p}$ and α explicit as long as possible, in order to show and analyze their coupling while the dependence on the material co-ordinate ρ, X_i and time t will be omitted, when ambiguities do not arise.

The strain measure a and $|\mathbf{r},_\rho|$ are strictly positive as a consequence of the regularity assumed for α and $\mathbf{p}$ and it is no longer necessity to introduce $|\mathbf{r},_\rho| > 0$ as an independent assumption. Furthermore, it can be observed that all the deformation parameters α, $\alpha,_\rho$ and $\mathbf{h}_{p},_\eta(\alpha)$ contribute to the local strain

$$\mathbf{r},_\rho = \mathbf{h}_{p},_\eta(\alpha)\alpha,_\rho \tag{14}$$

from which the other quantities a, $\mathbf{e}$ and $\bar{e}$ can be easily deduced. The notation $\mathbf{h}_{p},_\eta(\alpha)$, adopted here and in the sequel, means that the derivative of $\mathbf{h}_p$ with respect to its independent variable η is evaluated for $\eta = \alpha(\rho;t)$.

The velocity of the cable particles expressed by means of the kinematics descriptors, assumes the form

$$\dot{\mathbf{r}} = \mathbf{h}_{p},_\eta(\alpha)\dot{\alpha} + \dot{\mathbf{h}}_p(\alpha), \tag{15}$$

where the first term expresses the velocity due only to α and occurring even when the path $\mathbf{h}_p$ does not move, while the second term is due to the transport provided by the motion of the path $\mathscr{H}$. The acceleration assumes the following more complex form:

$$\ddot{\mathbf{r}} = \mathbf{h}_{p},_\eta(\alpha)\ddot{\alpha} + \mathbf{h}_{p},_{\eta\eta}(\alpha)\dot{\alpha}^2 + 2\dot{\mathbf{h}}_{p},_\eta(\alpha)\dot{\alpha} + \ddot{\mathbf{h}}_p(\alpha), \tag{16}$$

where the first two terms account for the body acceleration due to the motion along the fixed curved path, the third term represents the Coriolis acceleration and the last term is due to the path transport.

3. Balance conditions

In order to achieve a global balance condition for the system, it is assumed that the body consists of a simple hyperelastic material (Truesdell and Noll, 1965) and, consequently, at every material point the positive real valued function $w(X_i;\mathbf{C})$ is defined and describes the density of elastic potential energy by starting from the local strain measure $\mathbf{C} = (\nabla\mathbf{p})^T\nabla\mathbf{p}$. This energy tends to infinity when $|\nabla\mathbf{p}|$ or $\det\nabla\mathbf{p}$ tends to zero or to infinity. Once w is known, it is possible to derive the measure of the active stress furnished by the first and second Piola–Kirchhoff stress tensors, respectively denoted by $\mathbf{S}$ and $\bar{\mathbf{S}}$, by means of the following relations:

$$\mathbf{S}(X_i;\nabla\mathbf{p}) = \nabla\mathbf{p}\bar{S}(X_i;\nabla\mathbf{p}) = \nabla\mathbf{p}\frac{\partial w(X_i;C)}{\partial C}. \tag{17}$$

Even if the geometry of the cable makes it convenient to describe the kinematics using a uni-dimensional model, in real cases, its transverse dimension may equally play an important role in the interaction between the components, despite its smallness, and this aspect can be accounted by introducing some constitutive prescriptions for the region $\mathscr{G} \subset \mathscr{B}$ occupied by the tunnel.

The cable is hyperelastic in the sense that a positive real valued function $\omega(\rho;a)$ is defined and furnishes the elastic potential energy per unit length, measured in the reference configuration, by starting from the local deformation measure, described at each point ρ by $\mathbf{e} = a\bar{\mathbf{e}}$. In particular, the requirement of frame indifference for ω leads to the conclusion that ω can depend on a only. As previously mentioned, it is accepted that ω may diverge to infinity when its argument a tends to zero or infinity. The derivative $t(\rho;a)$ of $\omega(\rho;a)$ with respect to a denotes the intensity of the internal force produced in the cable by the strain and the derivative with respect to $\mathbf{e}$ provides a vector $\mathbf{t}(\rho;\mathbf{e})$, tangent to the cable path, which completely describes this force, i.e.

$$\mathbf{t}(\rho;\mathbf{e}) = t(\rho;a)\bar{\mathbf{e}}(\rho) = \omega_{,a}(\rho;a)\bar{\mathbf{e}}(\rho). \tag{18}$$

The system description is completed by assigning two positive scalar functions $m_0(X_i){:}\Omega \to \mathbb{R}^+$ and $\mu_0(\rho){:}I \to \mathbb{R}^+$ to describe mass density on the body and on the cable, in the reference configuration. The function m_0 is a mass per unit volume while the function μ_0 corresponds to a mass per unit length multiplied by $|\mathbf{R}_{,\rho}|$ so that its integral with respect to ρ furnishes the total mass of the cable.

With regard to the forces acting on the system, it is assumed that the body undergoes the mass forces $\mathbf{b}(X_i;\mathbf{p})$ acting on Ω and the contact forces $\mathbf{f}(X_i;\mathbf{p},\nabla\mathbf{p})$ acting on the portion $\partial\Omega_s$ of the boundary of Ω (Ciarlet, 1988), while the points lying on the remaining boundary portion $\partial\Omega_u$ cannot move and maintain the position occupied in the reference configuration. The cable is subjected to a generic distribution of mass forces $\beta(\rho;\mathbf{p},\alpha)$. As previously, β is a force per unit mass multiplied by $|\mathbf{R}_{,\rho}|$.

The balance equation is written in its weak form, on the basis of the Lagrange–D'Alembert principle (Truesdell and Toupin, 1960), as follows[1]:

$$\langle \mathbf{S}(\nabla\mathbf{p}),\nabla\hat{\mathbf{p}}\rangle_\Omega + \langle m_0(\ddot{\mathbf{p}} - \mathbf{b}(\mathbf{p})),\hat{\mathbf{p}}\rangle_\Omega - \langle \mathbf{f}(\nabla\mathbf{p}),\hat{\mathbf{p}}\rangle_{\partial\Omega_s} + \langle \mathbf{t}(\mathbf{r}_{,\rho}),\hat{\mathbf{r}}_{,\rho}\rangle_{I_\rho} + \langle \mu_0(\ddot{\mathbf{r}} - \beta),\hat{\mathbf{r}}\rangle_{I_\rho} = \varnothing \ \forall(\hat{\mathbf{p}},\hat{\mathbf{r}}) \in U;\ \forall t \in [0,\infty). \tag{19}$$

The form given to the principle tacitly represents an assumption on the mechanical behaviour of the system. In the case considered, it implies that no virtual power is joined to interaction forces between cable and body, neither at the anchorages or along the curve. It is also observed that at this stage, the balance condition must hold independently from the existence of the cable constraint. It is not within the aim of this paper to analyze questions regarding the existence of the solution and it is simply assumed that force, body stress and cable tractions permit defining the previous duality relations on the space U of admissible deformations $(\mathbf{p},\mathbf{r})$ defined on $\Omega \times I_\rho$.

In the case examined, where the cable undergoes the constraint previously described, the only admissible deformations are expressed by the couple $(\mathbf{p},\alpha) \in V$ and the test functions will be denoted by $\hat{\mathbf{p}}(X_i){:}\Omega \to \mathbb{R}^3$ and $\hat{\alpha}(\rho){:}I_\rho \to I_\rho$. The admissible variations $\hat{\mathbf{r}}$ of the deformation $\mathbf{r}$ and the admissible variations $\hat{\mathbf{r}}_{,\rho}$ of its derivative $\mathbf{r}_{,\rho}$ must be evaluated for $(\hat{\mathbf{p}},\hat{\alpha})$ lying on the space tangent to the constraint in correspondence of the generic state $(\mathbf{p},\alpha)$ existing at the considered instant t. Therefore, the variations $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}_{,\rho}$ can be deduced by linearizing Eqs. (13) and (14) and assume the following forms:

$$\hat{\mathbf{r}} = \mathbf{h}_{\hat{p}}(\alpha) + \mathbf{h}_{\rho,\eta}(\alpha)\hat{\alpha} \tag{20}$$

and

$$\hat{\mathbf{r}}_{,\rho} = \mathbf{h}_{\hat{p},\eta}(\alpha)\alpha_{,\rho} + \mathbf{h}_{p,\eta}(\alpha)\hat{\alpha}_{,\rho} + \mathbf{h}_{p,\eta\eta}(\alpha)\alpha_{,\rho}\hat{\alpha}, \tag{21}$$

where it can be noted that the deformation does not only involve the derivative $\nabla\hat{\mathbf{p}}$ and $\hat{\alpha}_{,\rho}$ but it involves also the slipping $\hat{\alpha}$ of the cable along the curve, mainly as a consequences of the path curvature.

[1] The crochet $\langle .,. \rangle$ denotes duality between spaces of vector-valued functions definite on the same domain. The expressions $\langle \mathbf{f},\mathbf{g}\rangle_\Omega$ and $\langle \mathbf{f},\mathbf{g}\rangle_{I_\rho}$ coincide with $\int_\Omega \mathbf{f}\cdot\mathbf{g}\,\mathrm{d}\Omega$ and $\int_{I_\rho} \mathbf{f}\cdot\mathbf{g}\,\mathrm{d}\rho$ for integrable functions products.

The previous balance condition becomes a linear form with respect to $(\hat{\mathbf{p}},\hat{\alpha})$; the part concerning the body remains the same while the part concerning the cable provides new duality relations between the kinematical entities chosen to describe the constrained deformation of the cable and dynamical entities. The cable traction $\mathbf{t}$ depends on $\nabla\mathbf{p}$, α, $\alpha,_\rho$ as a consequence of Eq. (14) and the term related to the virtual cable strain is decomposed in the following manner:

$$\langle\mathbf{t},\hat{\mathbf{r}},_\rho\rangle_{I_\rho} = \langle\alpha,_\rho\mathbf{t},\mathbf{h},_{\hat{p}},_\eta(\alpha)\rangle_{I_\rho} + \langle\mathbf{t}\cdot\mathbf{h}_\rho,_\eta(\alpha),\hat{\alpha},_\rho\rangle_{I_\rho} + \langle\alpha,_\rho\mathbf{t}\cdot\mathbf{h}_\rho,_{\eta\eta}(\alpha),\hat{\alpha}\rangle_{I_\rho}. \tag{22}$$

The vector $\mathbf{h}_{\hat{p}},_\eta(\alpha)$ which is a virtual deformation of $\mathbf{H},_\eta(\alpha)$ due to the body deformation $\hat{\mathbf{p}}$ only (α fixed), is put in duality, i.e. furnishes virtual power, with respect to the force $\mathbf{t}$ weighted by $\alpha,_\rho$, the scalar value $\hat{\alpha},_\rho$ describes a virtual strain on the fixed path $\mathbf{h}_p$ and produces virtual power with respect to the scalar quantities $\mathbf{t}\cdot\mathbf{h}_p,_\eta(\alpha) = t|\mathbf{h}_p,_\eta(\alpha)|$ provided by the scalar product of two parallel vectors, while $\hat{\alpha}$, which describes a virtual slipping along the fixed path $\mathbf{h}_p$, furnishes virtual power with respect to the scalar quantity $\alpha,_\rho\mathbf{t}\cdot\mathbf{h}_p,_{\eta\eta}(\alpha)$ where $\mathbf{h}_p,_{\eta\eta}(\alpha)$ is generally not parallel to $\mathbf{t}$ as a consequence of the path curvature. The dualities determined by the other term involving the cable are the following

$$\langle\mu_0(\ddot{\mathbf{r}}-\beta),\hat{\mathbf{r}}\rangle_{I_\rho} = \langle\mu_0(\ddot{\mathbf{r}}-\beta),\mathbf{h}_{\hat{p}}(\alpha)\rangle_{I_\rho} + \langle\mu_0(\ddot{\mathbf{r}}-\beta)\cdot\mathbf{h}_\rho,_\eta(\alpha),\hat{\alpha}\rangle_{I_\rho}, \tag{23}$$

where the former term is related to the path transport due to the body deformation $\hat{\mathbf{p}}$ while the latter is related to the slip $\hat{\alpha}$ and acts on the components of the mass forces along the tangent direction.

The corresponding balance condition relevant to the angular momentum of momentum is not reported because it does not furnish additional information, recalling that $\bar{\mathbf{S}}$ is, in this case, symmetric as a consequence of the existence of the strain energy density $w(X_i;C)$.

The expression given to the D'Alembert principle does not evidence some local aspects of the interaction arising between cable and body and, in particular, the absence of friction or other internal tangential forces along the path. An attempt to furnish a local interpretation of the global balance relation previously postulated, may be developed as follows.

If the reduced set of test functions $\hat{\mathbf{p}}$ whose support is contained in the domain $\bar{\Omega}$ is considered and $\hat{\alpha}=0$, then the generalized Gauß theorem leads to the condition

$$-\mathrm{Div}\,\mathbf{S} + m_0(\ddot{\mathbf{p}}-\mathbf{b}) = 0 \tag{24}$$

which must hold, in a generalized sense, on the internal points of the body less the points lying on the curve $\mathscr{H}$.

On the other hand, if those variations involving only $\hat{\alpha}$ are considered (i.e. $\hat{\mathbf{p}}=0$), the global form is equivalent to the condition

$$[-\mathbf{t},_\rho + \mu_0(\ddot{\mathbf{r}}-\beta)]\cdot\mathbf{h}_p,_\eta(\alpha) = 0 \tag{25}$$

so that the component of the internal cable force in the direction of the tangent of the cable path must be balanced by the component of β and $\ddot{\mathbf{r}}$ in the same direction.

The set of test functions $\hat{\mathbf{p}}$ defined on a compact support containing the tunnel $\mathscr{G}$ and, consequently, the curve $\mathscr{H}$, is now considered. In particular, this support consists of a tube containing $\mathscr{G}$ and having a constant section Γ. Therefore, the domain of the test functions $\hat{\mathbf{p}}:\Gamma\times(0,s)\rightarrow\mathbb{R}^3$ is provided by the cartesian product $\Gamma\times[0,s]$. Taking into account the previous balance condition and the relation $\mathbf{t},_\rho\cdot\bar{\mathbf{e}}=0$, it is possible to rewrite the weak balance condition by separating $\hat{\mathbf{p}}$ in the component $\hat{\mathbf{p}}_{\bar{\mathbf{e}}} = (\bar{\mathbf{e}}\otimes\bar{\mathbf{e}})\hat{\mathbf{p}}$ in the direction of the curve tangent and $\hat{\mathbf{p}}_n = \hat{\mathbf{p}} - \hat{\mathbf{p}}_{\bar{\mathbf{e}}}$ normal to the curve:

$$[-\mathrm{Div}\,\mathbf{S} + m_0(\ddot{\mathbf{p}}-\mathbf{b})]\cdot\bar{\mathbf{e}} = 0 \tag{26}$$

and

$$\langle -\text{Div}\,\mathbf{S} + m_0(\ddot{\mathbf{p}} - \mathbf{b}), \hat{\mathbf{p}}_n \rangle_\Gamma + [-\mathbf{t},_\rho + \mu_0(\ddot{\mathbf{r}} - \beta)] \cdot \hat{\mathbf{p}}_n = 0. \tag{27}$$

The component of body forces $m_0(\ddot{\mathbf{p}} - \mathbf{b})$ in the direction tangent to the curve are balanced by the stress divergence, such as occurs for the cable forces $\mu_0(\ddot{\mathbf{r}} - \beta)$ with respect to the derivative of the internal force $\mathbf{t}$ along the same direction; it is therefore now evident that the contact occurs without force in the direction tangent to the curve and that slipping is not prevented. On the other hand, a component of the stress is required on the surface of Γ for balancing cable forces in the plane normal to the path, even in the absence of external forces. This provides a distribution of forces which has a reactive nature for the cable; they arise due to the presence of the elastic constraint constituted by the path embedded in the deformable body and are not deducible from the cable deformation but from equilibrium only. In a similar way, the condition arising at the curve ends permits evidencing the forces that ensure equilibrium at the anchorages.

3.1. Linearized theory

In the numerical solution of the non-linear equations and in numerous situations of interest in engineering, particular attention is dedicated to the linear problem that can be derived from the previous theory by assuming that the motion develops in the neighbourhood of an assigned balanced configuration and by assuming that this neighbourhood is sufficiently small to make a formulation linearized with respect to the displacement, acceptable (*incremental problem*). Such a treatment makes the dependence of the cable deformation linear on the kinematical descriptors and this may also help comprehension of the problem. More precisely, in the examined case the reference configuration is chosen to coincide with the known balanced configuration and it is assumed that the norms of $\mathbf{u}(X_i;t) = \mathbf{p} - \mathbf{P}$, $\xi(\rho;t) = \alpha - \rho$ and their spatial derivatives are smaller than a parameter ϵ; the linear theory developed is such as to coincide with the exact theory when $\epsilon \to 0$ and such as to differ from the exact theory within an error bounded by ϵ^2.

The linearization of the cable deformation Eq. (13) furnishes the following relation, linear with respect to $\mathbf{u}$ and ξ ($\mathbf{h}_u = \mathbf{h}_p - \mathbf{H}$):

$$\mathbf{r} = \mathbf{R} + \mathbf{h}_u(\rho) + \mathbf{R},_\rho \xi + o(\epsilon), \tag{28}$$

where the term $\mathbf{h}_{u},_\eta \xi$, proportional to ϵ^2, has been neglected. The written $\mathbf{h}_u(\rho)$ denotes that the displacement field $\mathbf{h}_u$, definite on the set I_η of the curvilinear abscissa of the curve $\mathscr{H}$, is evaluated at $\xi = 0$ and, therefore, at $\eta = \alpha = \rho$. From the previous relation, the deformation expression can be deduced in the form

$$\mathbf{r},_\rho \simeq \mathbf{R},_\rho + \mathbf{h}_{u},_\eta(\rho) + \xi,_\rho \mathbf{R},_\rho + \xi \mathbf{R},_{\rho\rho}. \tag{29}$$

Three terms contribute to the cable strain. These are related to the path strain $\mathbf{h}_{u},_\eta$, the cable stretch $\xi,_\rho$ along the tangent direction $\mathbf{R},_\rho$ and the cable slip ξ which furnishes a component normal to the curve and proportional to the path curvature.

In the examined configuration, a known stress field $\mathbf{T}_0(X_i)$ is present on the body ($\mathbf{T}_0$ denotes the Cauchy stress tensor) and a field of known traction force $\mathbf{t}_0(\rho) = t_0(\rho)\mathbf{E}(\rho)$ is present along the cable. The stresses on the body and the forces on the cable are balanced with each other and balance the external actions existing in the reference configuration, denoted by $\mathbf{b}_0$, $\mathbf{f}_0$ and β_0. The linearization of the materials constitutive laws in the neighbourhood of these stress states furnishes the following relations at the points (X_i) of the body and ρ of the cable

$$\bar{\mathbf{S}}(\nabla\mathbf{u}) \simeq \mathbf{T}_0 + \mathbb{C}\nabla\mathbf{u} \tag{30}$$

and

$$\mathbf{t}(\mathbf{h}_{u,\eta}(\rho),\xi,\xi_{,\rho}) \simeq t_0\mathbf{E} + c(a-1)\mathbf{E} + t_0(\bar{\mathbf{e}} - \mathbf{E}), \tag{31}$$

where the fourth order tensor $\mathbb{C}$ acts on the symmetric part of $\nabla\mathbf{u}$ and denotes the derivative $2\partial\bar{\mathbf{S}}/\partial\mathbf{C}$ evaluated at $\mathbf{C}=\mathbf{I}$ while the scalar c denotes the derivative $\mathrm{d}t/\mathrm{d}a$ evaluated at $a=1$. To complete the description, it is necessary to determine the expressions of $(a-1)$ and $(\bar{\mathbf{e}}-\mathbf{E})$ when the deformation tends to zero. By recalling the definition of $a=|\mathbf{r}_{,\rho}|/|\mathbf{R}_{,\rho}|$ and linearizing, it becomes possible to deduce the following expression:

$$a-1 \simeq [\mathbf{h}_{u,\eta}(\rho) + \xi_{,\rho} + \mathbf{R}_{,\rho} + \xi\mathbf{R}_{,\rho\rho}]\cdot\mathbf{E}/|\mathbf{R}_{,\rho}| = (e-\mathbf{E})\cdot\mathbf{E}, \tag{32}$$

where it can be observed that, in the linear theory, stretch is furnished by the tangential component of the unit vector deformation. However both the body deformation, the slip and its derivative contribute to $(a-1)$, even if the contribution due to slip appears only if the parametrization is not normal. The previous expression can be made more explicit by introducing the notations $\mathbf{v}^t=(\mathbf{E}\otimes\mathbf{E})\mathbf{v}$ and $\mathbf{v}^n=(\mathbf{I}-\mathbf{E}\otimes\mathbf{E})\mathbf{v}$ to respectively denote the components of a vectorial field $\mathbf{v}$ on I_ρ in the tangent direction and the component lying on the plane orthogonal to the tangent. This leads to the following equivalent expression of Eq. (32):

$$a-1 \simeq \left(\left|\mathbf{h}^t_{u,\eta}(\rho)\right| + |\mathbf{R}_{,\rho}|\xi_{,\rho} + \left|\mathbf{R}^t_{,\rho\rho}\right|\xi\right)/|\mathbf{R}_{,\rho}|. \tag{33}$$

The infinitesimal variation of the tangent versor is described by the relation

$$\bar{\mathbf{e}} - \mathbf{E} \simeq [\mathbf{I}-\mathbf{E}\otimes\mathbf{E}][\mathbf{h}_{u,\eta}(\rho) + \xi_{,\rho}\mathbf{R}_{,\rho} + \xi\mathbf{R}_{,\rho\rho}]/|\mathbf{R}_{,\rho}| = [\mathbf{I}-\mathbf{E}\otimes\mathbf{E}](\mathbf{e}-\mathbf{E}). \tag{34}$$

In this case, it is the component of $(\mathbf{e}-\mathbf{E})$ normal to the path that acts. The components of $\mathbf{h}_{u,\eta}(\rho)$ and $\mathbf{R}_{,\rho\rho}$ orthogonal to the curve provide some effects while the term related to the derivative $\xi_{,\rho}$ is annihilated by the projector $\mathbf{I}-\mathbf{E}\otimes\mathbf{E}$, so that only the body deformation and cable slipping may provide a variation of orientation of the traction force. By introducing the previous notation, the expression assumes the form:

$$\bar{\mathbf{e}} - \mathbf{E} \simeq \left(\mathbf{h}^n_{u,\eta}(\rho) + \xi\mathbf{R}^n_{,\rho\rho}\right)/|\mathbf{R}_{,\rho}|. \tag{35}$$

The expressions obtained for kinematical and dynamical quantities in the linear theory can be used for writing the balance condition in terms of the unknown functions $\mathbf{u}$ and ξ. By recalling that the initial stresses are balanced, it is possible give the following form to the problem:

$$\begin{aligned}
&\langle(\nabla\mathbf{u}\mathbf{T}_0 + \mathbb{C}\nabla\mathbf{u}),\nabla\hat{\mathbf{u}}\rangle_\Omega + \langle m_0(\ddot{\mathbf{u}} - \mathbf{b} + \mathbf{b}_0),\hat{\mathbf{u}}\rangle_\Omega - \langle\mathbf{f}-\mathbf{f}_0,\hat{\mathbf{u}}\rangle_{\partial\Omega_S} + \langle(t_0\big(\mathbf{h}^n_{u,\eta}(\rho)\\
&\quad + \xi\mathbf{R}^n_{,\rho\rho}\big)/|\mathbf{R}_{,\rho}|,\left(\mathbf{h}^n_{\hat{u},\eta}(\rho) + \hat{\xi}\mathbf{R}^n_{,\rho\rho}\right)\rangle_{I_\rho} + \langle c\left(\left|\mathbf{h}^t_{u,\eta}(\rho)\right| + \xi_{,\rho} + \left|\mathbf{R}^t_{,\rho\rho}\right|\xi\right)/|\mathbf{R}_{,\rho}|,\left|\mathbf{h}^t_{\hat{u},\eta}(\rho)\right| + |\mathbf{R}_{,\rho}|\hat{\xi}_{,\rho}\\
&\quad + \left|\mathbf{R}^t_{,\rho\rho}\right|\hat{\xi}\rangle_{I_\rho} + \langle\mu_0(\ddot{\mathbf{h}}_u(\rho) + \ddot{\xi}\mathbf{R}_{,\rho} - \beta + \beta_0),(\mathbf{h}_{\hat{u}}(\rho) + \hat{\xi}\mathbf{R}_{,\rho})\rangle_{I_\rho}\\
&\quad = \varnothing \;\; \forall(\hat{\mathbf{u}},\hat{\xi}) \in V;\; \forall t \in [0,\infty).
\end{aligned} \tag{36}$$

The linear formulation furnishes a bilinear form $q(\mathbf{u},\xi;\hat{\mathbf{u}},\hat{\xi}): V\times V\to\mathbb{R}$ that can be decomposed as the sum $q=q_f+q_m+q_c+q_t$ where q_f is related to the external force $\mathbf{f}$, $\mathbf{b}$, β; q_m contains the inertial terms m_0 and μ_0; q_c collects the terms related to the constitutive functions $\mathbb{C}$ and c, and q_t collects the terms related to the balanced stresses $\mathbf{T}_0$ and t_0 existing in the reference configuration. The bilinear form q_c is usually positive definite while q_t may be non positive definite and can produce unstable motions in the energy norm, i.e. with respect to the Liapunov functional

$$V(\mathbf{u},\xi;t) = \langle m_0\dot{\mathbf{u}},\dot{\mathbf{u}}\rangle_\Omega + \langle \mu_0\dot{\mathbf{r}},\dot{\mathbf{r}}\rangle_{I_\rho} + \langle \mathbb{C}\nabla\mathbf{u},\nabla\mathbf{u}\rangle_\Omega + \langle c(a-1),(a-1)|\mathbf{R},_\rho|\rangle_{I_\rho} + \langle \nabla\mathbf{u}T_0,\nabla\mathbf{u}\rangle_\Omega$$
$$+ \langle t_0(\bar{e}-\mathbf{E}),(\bar{e}-\mathbf{E})|\mathbf{R},_\rho|\rangle_{I_\rho}. \tag{37}$$

4. Massless homogeneous cable

It is now assumed that the mass μ_0 of the cable is null, so that the previous external and inertial forces on the cable do not provide effects. It is also assumed that the cable is homogeneous and, consequently, the function $\omega(a)$ expressing the elastic energy per unit length is the same for all the points ρ. Furthermore, it is assumed that $\omega(a)$ is convex and $t(a)$ is thus monotone.

This situation has particular relevance because often in engineering applications, the mechanical characteristics of the cable are constant along it and so much better than the mechanical characteristics of the three dimensional body as to permit adopting cables having a mass negligible with respect to the body mass. Qualitative considerations of the behaviour of this system with massless cable can be deduced from the analysis of the static case discussed in Dall'Asta, 1995 and are not repeated here. The aim of this paragraph is to evidence the link and the differences between the more general theory of the previous paragraph and this particular case.

The local balance equation of the cable in the direction tangent to its path, determined in advance integrating by part in the case of test functions $\hat{\alpha}$, now leads to the following condition:

$$\mathbf{t},_\rho \cdot \mathbf{e} = (\omega,_{aa}a,_\rho\bar{\mathbf{e}} + \omega,_a\bar{\mathbf{e}},_\rho)\cdot\bar{\mathbf{e}} = 0 \tag{38}$$

from which, taking into account that $\bar{\mathbf{e}}$ is a unit vector and ω is convex, it can be deduced that the condition $a,_\rho=0$ must also be verified. Therefore, the assumption of the absence of mass and homogeneity for the cable, together with the previous condition of frictionless contact implicitely expressed by the balance condition, immediatly leads to a reduction of the deformation space to its proper subspace in which the cable strain is homogeneous. This also permits the assertion that the potential elastic energy contained in the cable is constant along the cable, even if the internal force $\mathbf{t}$ may vary in direction, and can be deduced simply by multiplying the initial length by the energy density corresponding to the constant strain a.

The deformations subjected to the stronger constraint of homogeneous strain

$$a,_\rho(\rho;t) = 0 \tag{39}$$

will be analized in detail. Firstly, it follows from Eq. (39) that a must coincide with its mean value and the latter may be deduced from the ratio between the total length

$$l_p = \gamma_p(s) = \int_0^s |\mathbf{h}_p,_\zeta|\mathrm{d}\zeta \tag{40}$$

of the path in the deformed configuration and the total length

$$L = \Lambda(s) = \int_0^s |\mathbf{H},_\zeta|\mathrm{d}\zeta \tag{41}$$

of the path in the reference configuration, where both the quantities can be derived from the body configurations only. In conclusion, a does not depend on ρ and has the form

$$a(t) = \frac{l_p}{L}. \tag{42}$$

The functional dependence of the cable local strain on the global body deformation now becomes more explicit than in the general case (Eqs. (11) and (12)).

However, the mean aspect making the massless case substantially different from the case analyzed in the previous paragraph is that it is no longer necessary to introduce function $\alpha(\rho;t)$ to describe the system deformation because the slipping of the cable is not affected by any external force and the cable deformation can be completely determined from the the body deformation only, as will be shown in the sequel. In particular, in order to provide the final position of each cable point ρ, it is recalled that in paragraph 2, the function $\gamma_p(\eta)$ was introduced. This measures the length of the path linked to the body from the material point $(H_i(0))$ to $(H_i(\eta))$, in the deformed configuration, its derivative satisfies the inequality $\gamma_{p,\eta} > 0$ and the function is invertible. The homogeneity of the strain ensures the equality of the two ratios $\gamma_p(\eta)/l_p$ and $\Lambda(\rho)/L$ for each cable material particle ρ lying at the point $\mathbf{h}_p(\eta)$ of the curve $\mathscr{H}$, so that the function $\lambda_r(\rho)$ now depends on $\mathbf{p}$ only and coincides with the function $\lambda_p(\rho) = a(t)\Lambda(\rho)$, which provides the following relation:

$$\eta = \alpha(\rho;t) = \left(\gamma_p^{-1} \bigcirc \lambda_p\right)(\rho) \tag{43}$$

between the cable points and the body curve points. It becomes possible to reconstitute the position of each cable point by means of the relation

$$\mathbf{r}(\rho;t) = \left(\mathbf{h}_p \bigcirc \gamma_p^{-1} \bigcirc \lambda_p\right)(\rho) \quad \rho \in [0,s] \tag{44}$$

and the complete kinematic description of the system is finally obtained on the basis of the function $\mathbf{p}$ only.

The global balance conditions may now be stated by following the previous process and assuming as admissible deformations only those providing homogeneous strain in the cable. It would be possible to write the equilibrium condition simply by replacing α with the previously defined function Eq. (43), but this is not convenient because it is not an easy matter to derive this from the deformation $\mathbf{p}$. However, the use of γ_p^{-1} can be avoided altogether by evaluating the integral of the scalar product between $\mathbf{t} = t\bar{\mathbf{e}}$ and $\mathbf{r}_{,\rho} = (\mathbf{h}_p \bigcirc \gamma_p^{-1} \bigcirc \lambda_p)_{,\rho}$ directly with respect to the body curve parameter η instead of with respect to the cable material co-ordinate ρ, taking into account that $\lambda_{p,\rho}\,\mathrm{d}\rho = \gamma_{p,\eta}\,\mathrm{d}\eta$. The form of the balance condition particularly enjoys this position while the cable deformation, that does not explicitly appear, can be equally reconstituted once $\mathbf{p}$ is known, via γ_p^{-1}, even if this is not very interesting because the most interesting design parameters are the cable internal force and strain which are both related to a only. The balance condition at the instant t in which the deformation $\mathbf{p}$ is present, has the following expression:

$$\begin{aligned} &\langle \mathbf{S}, \nabla\hat{\mathbf{p}} \rangle_\Omega + \langle m_0(\ddot{\mathbf{p}} - \mathbf{b}), \hat{\mathbf{p}} \rangle_\Omega - \langle \mathbf{f} \cdot \hat{\mathbf{p}} \rangle_{\partial\Omega_s} + t(a)\int_0^s \bar{\mathbf{g}}_p \cdot \mathbf{h}_{\hat{p},\eta}\,\mathrm{d}\eta = 0 \\ &\forall \hat{\mathbf{p}} \in V; \forall t \in [0,\infty). \end{aligned} \tag{45}$$

The latter makes it evident that the dual entity of the force t, constant along the cable, is also a constant quantity which coincides with

$$\hat{a} = \frac{1}{L}\int_0^s \bar{\mathbf{g}}_p \cdot \mathbf{h}_{\hat{p},\eta}\,\mathrm{d}\eta \tag{46}$$

expressing the cable strain variation provided by $\hat{\mathbf{p}}$ on the space tangent to the constraint at $\mathbf{p}$ and the term concerning the cable can also be simply furnished by the product $t\hat{a}L$. The latter is a product of functionals of the body deformation and is now not required to integrate quantities varying along the cable, like the vectors $\mathbf{t}$ and $\mathbf{r},_\rho$ in Eq. (22).

4.1. Linearized theory

As in the previous case, here again it is interesting to state the linear equations describing infinitesimal motions near an assigned balanced configuration. The procedure follows the same steps, so that the reference configuration is chosen so as to coincide with the known configuration, the deformation is denoted by means of the position $\mathbf{u}(X_i;t)=\mathbf{p}-\mathbf{P}$, $\xi(\rho;t)=\alpha-\rho$ and the linearized theory is obtained by assuming that $\mathbf{u}$, ξ and their derivatives are small. It is not necessary to introduce the function γ_p^{-1} and it is possible to continue independently from the results of the previous paragraph, starting directly from Eq. (45). In the range of a linear approximation, the module t of the force $\mathbf{t}$ assumes the expression

$$t \simeq t_0 + c(a-1) \tag{47}$$

with

$$(a-1) \simeq \frac{1}{L}\int_0^s \mathbf{h}_{u,\eta} \cdot \mathbf{G}\,\mathrm{d}\eta \tag{48}$$

while the linearization of the variation of the functional $\hat{a}$ in the neighbourhood of the origin furnishes

$$\hat{a} \simeq \frac{1}{L}\int_0^s \mathbf{G} \cdot \mathbf{h}_{\hat{u},\eta}\,\mathrm{d}\eta + \frac{1}{L}\int_0^s (\mathbf{I} - \mathbf{G}\otimes\mathbf{G})\frac{\mathbf{h}_{u,\eta}\cdot\mathbf{h}_{\hat{u},\eta}}{|\mathbf{H},_\eta|}\,\mathrm{d}\eta. \tag{49}$$

By using the same notation used previously for denoting tangential and normal components along the path traced by the cable, it is possible to derive the following final formulation:

$$\begin{aligned}
&\langle \nabla\mathbf{u}T_0 + \mathbb{C}\nabla\mathbf{u}, \nabla\hat{\mathbf{u}}\rangle_\Omega - \langle m_0(\ddot{\mathbf{u}} - \mathbf{b} + \mathbf{b}_0), \hat{\mathbf{u}}\rangle_\Omega - \langle \mathbf{f} - \mathbf{f}_0, \hat{\mathbf{u}}\rangle_{\partial\Omega_s} + \frac{c}{L}\int_0^s \left|\mathbf{h}^t_{u,\eta}\right|\mathrm{d}\eta \int_0^s \left|\mathbf{h}^t_{\hat{u},\eta}\right|\mathrm{d}\eta \\
&\quad + t_0\int_0^s \frac{\mathbf{h}^n_{u,\eta}\cdot\mathbf{h}^n_{\hat{u},\eta}}{|\mathbf{H},_\eta|}\,\mathrm{d}\eta \\
&\quad = \varnothing \;\; \forall(\hat{\mathbf{u}},\hat{\xi}) \in V;\ \forall t \in [0,\infty).
\end{aligned} \tag{50}$$

5. Application

The proposed formulation can be used in analyzing structural problem, usually concerning rods, plates or shells, by introducing suitable constrained kinematical models for the solid. The following application examines a system consisting of a rectangular plate containing a cable lying on the middle plane and parallel to one side, in order to analyze the consequences of cable stretching on vibration modes, in the range of small deformations (Fig. 2). This technique is used to reduce tensile stresses caused by bending actions on the plate and the case considered permits demonstrating some qualitative aspects despite its simplicity. The modal analysis of this system is interesting because it permits the evaluation of the real traction force existing in the cable via non-destructive tests.

The plate is a prismatic solid occupying the region $\Omega=\{X_\alpha\mathbf{A}_\alpha+X_3\mathbf{A}_3;\ \alpha=1,2;\ X_\alpha\in[0,D_1]\times[0,D_2];\ X_3\in[-d/2,d/2]\}$ in the reference configuration. The curve $\mathscr{H}=\{X_1=\eta,\ X_2=s,\ X_3=0;\ \eta\in[0,D_1]\}$ is defined on the body and traces the following path in the reference configuration

$$\mathbf{H}(\eta)=H_i(\eta)\mathbf{A}_i=\eta\mathbf{A}_1+s\mathbf{A}_2\ \eta\in[0,D_1]. \tag{51}$$

Its derivative is furnished by $\mathbf{H}_{,\eta}(\eta)=\mathbf{G}(\eta)=\mathbf{A}_1$. In the reference configuration, an internal force t_0 acts on the cable. The parameter s, defining the position of the cable, is often designed for balancing, or reducing, the bending stress induced by external action. The following applications analyze the simplified situation in which the stress field on the body, produced by external forces and interaction with the cable, consists of an uniform compressive stress $\mathbf{T}_0=-t_0/\mathrm{d}D_2(\mathbf{A}_1\otimes\mathbf{A}_1)$; this situation may occur if the plate is a portion of a larger structure, e.g. a beam web between stiffenings, or a rigid apparatus providing equilibrium is posed at the sides $X_1=0,\ X_1=D_1$.

The infinitesimal motions are analyzed describing the behaviour of the plate by means of the following displacement field, valid for a transversely isotropic material internally constrained by the condition $\mathrm{Sym}(\nabla\mathbf{u})\cdot(\mathbf{A}_i\otimes\mathbf{A}_3)=0\ (i=1,2,3)$ (see Podio-Guidugli, 1989):

$$\mathbf{u}(X_\alpha,\zeta;t)=-X_3\nabla v(X_\alpha;t)+v(X_\alpha;t)\mathbf{A}_3, \tag{52}$$

where v denotes the transversal displacements of the middle plane $(X_1,X_2,0)$ and its gradient is the vector $v_{,\alpha}\mathbf{A}_\alpha$ (repeated indices denote summation). The expressions of the displacement gradient and its symmetric part are the following:

$$\nabla\mathbf{u}=-X_3\nabla\nabla v+(\mathbf{A}_3\otimes\nabla v-\nabla v\otimes\mathbf{A}_3) \tag{53}$$

and

$$\mathrm{Sym}(\nabla\mathbf{u})=-X_3\nabla\nabla v, \tag{54}$$

where $\nabla\nabla v=v_{,\alpha\beta}(\mathbf{A}_\alpha\otimes\mathbf{A}_\beta)$.

Such a displacement field maps $\mathbf{H}$ into the curve

$$\mathbf{H}(\eta)+\mathbf{h}_u(\eta)=\eta\mathbf{A}_1+s\mathbf{A}_2+v(\eta,s;t)\mathbf{A}_3 \tag{55}$$

while the term $\mathbf{h}_{u,\eta}(\eta)$ assumes the following form:

$$\mathbf{h}_{u,\eta}(\eta)=v_{,1}(\eta,s;t)\mathbf{A}_3. \tag{56}$$

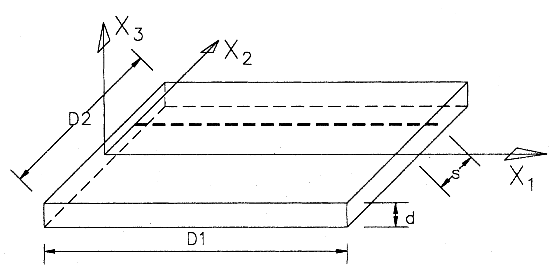

Fig. 2. Geometry of the plate with internal cable.

The material forming the plate is homogeneous; E is its normal elastic modulus in the plane X_1-X_2 and ν; the corresponding Poisson modulus. Furthermore, J denotes the geometric quantity $d^3/12$ and $B=EJ/(1-\nu^2)$. The displacement model considered provides the following expression for the bilinear form (50) ($\mathbf{h}_u$ was made explicit by Eq. (56)):

$$
\begin{aligned}
&B\int_0^{D_1}\int_0^{D_2}[v_{,11}\hat{v}_{,11}+v_{,22}\hat{v}_{,22}+2\nu(v_{,11}\hat{v}_{,22}+v_{,22}\hat{v}_{,11})+2(1-\nu)v_{,12}\hat{v}_{,12}]\mathrm{d}X_1\,\mathrm{d}X_2\\
&-t_0\frac{J}{dD_2}\int_0^{D_1}\int_0^{D_2}[v_{,11}\hat{v}_{,11}+v_{,12}\hat{v}_{,12}+\frac{h}{J}v_{,1}\hat{v}_{,1}]\mathrm{d}X_1\,\mathrm{d}X_2+\int_0^{D_1}\int_0^{D_2}m_0\,d\ddot{v}\hat{v}\,\mathrm{d}X_1\,\mathrm{d}X_2\\
&+t_0\int_0^{D_1}v_{,1}(\eta,s;t)\hat{v}_{,1}(\eta,s)\mathrm{d}\eta\\
&=\varnothing\,\forall\hat{v}\in V;\ \forall t\in[0,\infty).
\end{aligned}
\qquad (57)
$$

The lack of the term related to the cable stiffness c is due to the particular situation analyzed here, where the path is rectilinear and the kinematical internal constraint of the plate prevents axial strain in the middle plane.

The case of boundary conditions preventing displacements in the direction of $\mathbf{A}_3$ along the sides parallel to the X_2-axis is considered. The solution can be sought in the form $v=\exp(i\theta t)\phi(X_\alpha)$ where ϕ can be approximated by means of a sine series for X_1 and a Legendre series for X_2. The numerical results concern a plate characterized by the following parameters: $d=0.2$ m, $D_2=20$d, $E=2.5\times10^{10}$ N/m^2, $m_0/ED_2d=8.3\times10^{-8}$, $\nu=0.2$.

Figs. 3 and 4 report the first five periods obtained for different values of the cable force, expressed by the non-dimensional ratio $t*=t_0/ED_2d$, corresponding to the plate strain due to the compression. The results are reported in terms of ratio between the period $\tau=2\pi/\theta$ corresponding to $t*$ and the period τ_0 obtained in the absence of the cable. The results described in Fig. 3 have been obtained for a square plate with $D_1/D_2=1.0$ while the results described in Fig. 4 have been obtained for a rectangular plate with $D_1/D_2=2.0$. In both cases, two diagrams are reported; Fig. 3(a) and Fig. 4(a) refer to the case of $s=0.0$, i.e. to the case of a cable placed at the centre of the plate, Fig. 3(b) and Fig. 4(b) refer to the case of an eccentric cable located at $s=0.45D_2$.

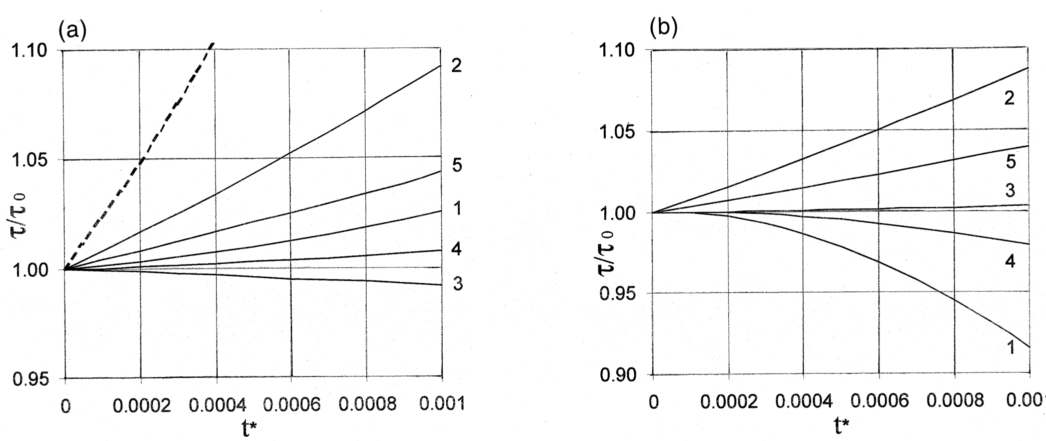

Fig. 3. Variation of eigenperiods versus cable traction for a square plate.

From the diagrams, it can be observed that the interaction between the stretched internal cable and the compressed plate can produce very different effects on modal properties. In particular, some eigenperiods decrease while other ones increase. Furtermore, the influence of the prestressing force does not provide significant influence for all eigenperiods and some present a small variation. Larger reductions of eigenperiods have been obtained in the case of an eccentric cable [Fig. 3(b) and Fig. 4(b)].

Such a different behavior can be explained by observing the two terms of Eq. (57) related to the cable force t_0. The first one, deriving from integration on the plate, is negative definite (every displacement field v implies $v_{,1}$ somewhere for the considered boundary conditions), so that it provides a reduction in the stiffness of the system. The second one, deriving from the cable, is only positive semi-definite and it provides a stiffening contribution only for those vibration modes involving displacement of the cable path. For each vibration mode, such a stiffning contribution, if it exists, can be larger or smaller than the negative effect produced by the compressive stress on the plate and described by th first negative term; the corresponding eigenperiod consequently results shorter or longer than the eigenperiod observed for the plate free from stress.

The dashed lines of Fig. 3(a) and Fig. 4(a) refer to the first eigenperiod measured in the different case where the internal cable is not present and the compressive stress $\mathbf{T}_0$ in the plate is produced by an external force or by the interaction with two parallel cables which run externally to the plate and are in contact with the plate at the anchorages only.

In this case, the period of the first vibration mode increases and the state of stress provides a more remarkable effect. This occurs because the positive semi-definite term related to the internal cable is now absent so that the system shows a totally different behavior and all the eigenperiods become as large as the stress increases, even if not all the eigenperiods are strongly influenced as the first one.

In conclusion, an internal slipping cable makes it possible to obtain a notable state of compressive stress on the plate avoiding unacceptable reduction in stiffness (and instability as limit case). The modal properties of the system however change and eigenperiods may enlarge or reduce. The prediction of such variations in not easy and depend, besides on the stress intensity, both on the shape of vibration modes and on the path of the cable.

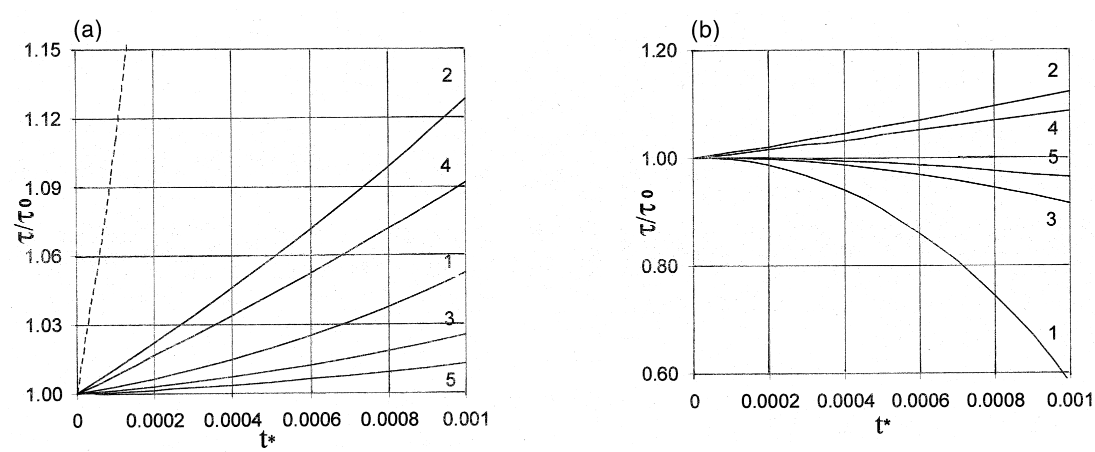

Fig. 4. Variation of eigenperiods versus cable traction for a rectangular plate with $D_1/D_2 = 2.0$.

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